Optimal routing and placement of orders in limit order markets

Rama CONT
Imperial College
London

Arseniy KUKANOV
Columbia University
New York

CFEM-GARP Joint Event and Seminar 05/01/13, New York
Modern trading is largely automated, institutional investors use algorithms to spread executions of large trades across time.

An execution of a small order is further optimized by selecting its type and trading venues for its placement.

- The same stock can be traded on a dozen of exchanges (in the U.S.) and many more alternative venues.
- Exchanges offer a variety of order types and qualifiers (more than 20 on NYSE) and an inter-exchange order routing logic.

Exchanges compete for order flow by offering liquidity rebates.

Market participants develop “smart order routers” to improve their execution quality (see Boehmer and Jennings (2007), Foucault and Menkveld (2008), Maglaras et al. (2012)).
Optimal trade execution

Traders are faced with various decisions:

- Optimal allocation across time: scheduling/splitting of trades into small orders
- Choosing an order type for each small batch of orders: limit, market, ...
- Choosing a trading venue (or venues) for submitting each small batch of orders

Most of the research literature has focused on optimal allocation across time: Bertsimas and Lo (1998), Almgren and Chriss (2000,...), Obizhaeva and Wang (2005), Schied et al. (2010),...

Main idea: tradeoff between price impact and volatility/price uncertainty. Does not use information on order books, order flow,..

Most importantly: this literature does not study the **execution risk** - the risk of not filling an order.
The literature on optimal order execution investigates how to reduce a cost of a large trade by splitting it into a sequence of orders.

- Obizhaeva and Wang (2005), Shreve et al. (2011): only market orders are used, executions are given by an order book shape function.
- Laruelle et al. (2009), Ganchev at al. (2010): numerical algorithms for order allocation across multiple dark pools.
The literature on optimal order execution investigates how to reduce a cost of a large trade by splitting it into a sequence of orders.

- Obizhaeva and Wang (2005), Shreve et al. (2011): only market orders are used, executions are given by an order book shape function.
- Laruelle et al. (2009), Ganchev at al. (2010): numerical algorithms for order allocation across multiple dark pools.

Solving the trade scheduling and the order placement problems in one setting requires restrictive assumptions on price and order book dynamics.

We follow a simpler, more tractable approach. It allows us to incorporate realistic features that matter for order placement decisions, while conserving analytical tractability.
The order placement problem

We consider here the **order placement problem**: Suppose you have decided now to send a batch of orders to the market.

- Order routing problem: how do you split the orders between different venues?
- How does this decision use the information on the market depth, speed of fill, etc?
- Choosing order type: limit order, market order or mix?

Here we will present a framework for studying this problem separately from the scheduling problem previously studied by Bertsimas and Lo, Almgren and Chriss, and others.

In principle: could be combined in a (too?) ambitious stochastic control framework but analytically untractable unless very restrictive assumptions are imposed...
Model ingredients

Objective: buy $S$ shares by time horizon $T$ - fixed or random time.

- Multiple venues $k = 1, \ldots, K$ are characterized at time 0 by:
  - Bid queue lengths $Q_k$
  - Liquidity “maker” rebates $r_k$ for exchange $k$
  - Liquidity “taker” fees $f_k$, bid-ask spread $h$

- Generally we can send market and limit orders to each venue.
- But market order execution is simpler and much more certain than limit order execution.
- To simplify: assume that a single market order is sent to the venue with the smallest fee $f$, its execution is immediate and certain.

Order placement strategy: $X = (M, L_1, \ldots, L_K)$
$M =$ volume of market orders, $L_k =$ limit orders placed in venue $K$
Problem: we want to buy $S$ shares by time $T$ at the lowest cost using multiple orders. Key tradeoff: price vs. execution probability.
Problem: we want to buy $S$ shares by time $T$ at the lowest cost using multiple orders. Key tradeoff: price vs. execution probability.
**Problem:** we want to buy $S$ shares by time $T$ at the lowest cost using multiple orders. **Key tradeoff:** price vs. execution probability.
Orders are submitted now, executions are observed later and depend on future order flows. Order sizes are not infinitesimal - partial fills are important.

How much will we fill?

\[ L_1 \]

\[ Q_1 \]
Orders are submitted now, executions are observed later and depend on future order flows. Order sizes are not infinitesimal - partial fills are important.

\[ C_1 \in [0, Q_1] \]
Orders are submitted now, executions are observed later and depend on future order flows. Order sizes are not infinitesimal - partial fills are important.

\[ C_1 \in [0, Q_1] \]

Some orders will fill:
\[ D_1 \geq 0 \]

How much will we fill?

Some orders will cancel:
\[ Q_1 \]
Orders are submitted now, executions are observed later and depend on future order flows. Order sizes are not infinitesimal - partial fills are important.

How much will we fill? 
Some orders will cancel: 
\( C_1 \in [0, Q_1] \)

Some orders will fill: 
\( D_1 \geq 0 \)

Define: \( \xi_1 = C_1 + D_1 \)

Our filled quantity:
\[
\min \left\{ \max \left( \xi_1 - Q_1, 0 \right), L_1 \right\} = \\
= \left( \xi_1 - Q_1 \right)_+ - \left( \xi_1 - Q_1 - L_1 \right)_+ 
\]
Orders are submitted now, executions are observed later and depend on future order flows. Order sizes are not infinitesimal - partial fills are important.

\[ \xi_1 = C_1 + D_1 \]

Our filled quantity:

\[ \min \{ \max(\xi_1 - Q_1, 0), L_1 \} = (\xi_1 - Q_1)_+ - (\xi_1 - Q_1 - L_1)_+ \]
Orders are submitted now, executions are observed later and depend on future order flows. Order sizes are not infinitesimal - partial fills are important.
A trader is given an execution horizon $T$ - known or random stopping time. At $t = 0$, the trader places $M$ market orders, $L_k$ limit orders on venues $k = 1, \ldots, K$. Between 0 and $T$: $\xi_k =$ bid queue outflows due to cancelation and executions in front or traders orders. Then the total execution quantity by the end of $[0, T]$ can be written as a function of future queue outflows:

$$A(X, \xi) = M + \sum_{k=1}^{K} ((\xi_k - Q_k)_+ - (\xi_k - Q_k - L_k)_+)$$
Execution cost

Transaction cost: total cost paid = benchmark cost computed using mid-quote price + execution cost relative to mid-quote:

\[(h + f)M - \sum_{k=1}^{K} (h + r_k)((\xi_k - Q_k)_+ - (\xi_k - Q_k - L_k)_+), \quad (1)\]

where \( h \) is a half of the bid-ask spread at time 0, \( f \) is the lowest available liquidity fee and \( r_k, k = 1, \ldots, K \) are liquidity rebates.
Execution risk

The total executed quantity is random:

$$A(X, \xi) = M + \sum_{k=1}^{K} ((\xi_k - Q_k)_+ - (\xi_k - Q_k - L_k)_+)$$

Idea: introduce a penalty for violations of target quantity $S$:

$$\lambda_u (S - A(X, \xi))_+ + \lambda_o (A(X, \xi) - S)_+$$

If $A(X, \xi) < S$, the trader has to purchase the remaining $S - A(X, \xi)$ shares at time $T$ with market orders. Adverse selection implies that conditionally on the event $\{A(X, \xi) < S\}$ prices have likely moved up and the transaction cost of market orders at time $T$ is higher than their cost at time $0$, i.e. $\lambda_u > h + f$.

If $A(X, \xi) > S$ the trader experiences buyer’s remorse - conditionally on this event prices have likely moved down and he could have achieved a better execution by sparing some of the market orders.
An optimal order placement is a vector \( X^\ast \in \mathbb{R}^{K+1}_+ \) solution of

\[
\min_{X \in \mathbb{R}^{K+1}_+} \mathbb{E}[v(X, \xi)] \quad \text{where} \quad (2)
\]

\[
v(X, \xi) := (h + f)M - \sum_{k=1}^{K} (h + r_k)((\xi_k - Q_k)_+ - (\xi_k - Q_k - L_k)_+) \\
+ \lambda_u (S - A(X, \xi)))_+ + \lambda_o (A(X, \xi) - S)_+
\]

is the sum of the execution cost and penalty for execution shortfall.

We will denote \( V(X) = \mathbb{E}[v(X, \xi)] \).
Existence of an optimal order allocation

Assume that:

- \( \lambda_o > h + \max_k \{r_k\} \) and \( \lambda_o > -(s + f) \): it is suboptimal to exceed the target quantity \( S \) regardless of fees and rebates.
- \( \min_k \{r_k\} + h > 0 \): limit orders still provide a better price, even after accounting for (possibly negative) rebates.

**Proposition** (Cont & AK, 2012) Under these assumptions, there exists an optimal allocation \( X^* = (M^*, L_1^*, ... L_K^*) \) which lies in

\[
C = \left\{ X \in \mathbb{R}_+^{K+1} \mid 0 \leq M \leq S, \ 0 \leq L_k \leq S - M, \ M + \sum_{k=1}^{K} L_k \geq S \right\}
\]

- Total order size may overflow target: \( M + \sum_{k=1}^{K} L_k \geq S \)
- But \( X^* \in C \) so it is never optimal to overflow target with a single type of order or single trading venue.
Soft vs hard constraints

- The penalty term for execution risk effectively implements a soft constraint for order sizes and focuses the search for an optimal order allocation to the set $C$.

- Once can also handle hard constraints, e.g. $M = 0$ or $\sum_{k=1}^{K} L_k = S$ but it is not clear whether they are 'optimal' in the sense of the tradeoff between execution cost and execution risk. The present formulation allows precisely to explore this point (more later).
When there is only one trading venue, the problem is to choose between limit and market orders.

**Proposition** Assume that the bid queue outflow $\xi$ has a continuous distribution $F$.

If $\lambda_u \leq \lambda_u = \frac{2h + f + r}{F(Q + S)} - (h + r)$, then the optimal allocation involves limit orders only $(M^*, L^*) = (0, S)$.

If $\lambda_u \geq \lambda_u = \frac{2h + f + r}{F(Q)} - (h + r)$, then the optimal allocation involves market orders only $(M^*, L^*) = (S, 0)$.

(3) If $\lambda_u \in (\lambda_u, \lambda_u)$, there exists an optimal split given by

$$M^* = S - F^{-1}\left(\frac{2h + f + r}{\lambda_u + h + r}\right) + Q, \quad L^* = F^{-1}\left(\frac{2h + f + r}{\lambda_u + h + r}\right) - Q.$$
In all cases, with a single trading venue $M^* + L^* = S$: no risk of exceeding target size.

- Tradeoff between shortfall risk and execution fees.
- As order size $S$ increases, a larger fraction $\frac{M^*}{S}$ is executed with market orders.
- Solution depends on full distribution $F$ of the bid queue outflow, not just its mean.
- The likelihood of a limit order execution on exchange $k$ is given by the probability $\mathbb{P}(\xi_k > Q_k)$, not by average values of $\xi_k$ or $Q_k$. 
Single trading venue: split between limit / market orders

**Figure**: Optimal limit order size for different values of urgency parameter $\lambda_u$. Parameter values: $Q = 2000$, $S = 1000$, $h = 0.02$, $r = 0.002$, $f = 0.003$. 
When multiple trading venues are available, dividing the target quantity among them provides better execution quality - it reduces the risk of not filling the order.

Sending too many orders leads to an undesirable possibility of overfulfilling the target size.

An optimal order allocation must balance these risks and take into account pricing parameters. We need a criterion for optimality.
Multiple trading venues: optimal order routing

**Proposition:** Assume that the bid queue outflows \((\xi_1, \ldots, \xi_K)\) have a continuous joint distribution \(F\) and for each exchange

\[
\Pr(\xi_k < Q_k + S) < 1 \quad \text{and} \quad \Pr(\xi_k < Q_k) < \frac{2h + f + r_k}{\lambda_u + h + r_k}.
\]

- An optimal order allocation involves market orders if

\[
\lambda_u \geq \frac{2h + f + \max_k \{r_k\}}{\Pr\left(\bigcap_k \{\xi_k \leq Q_k\}\right) - (h + \max_k \{r_k\})} \quad (3)
\]

- An optimal order allocation uses limit orders in exchange \(j\) if

\[
\Pr\left(\bigcap_{k \neq j} \{\xi_k \leq Q_k\} \mid \xi_j > Q_j\right) > \frac{\lambda_o - (h + r_j)}{\lambda_u + \lambda_o} \quad (4)
\]
Characterization of optimal allocation in terms of shortfall probabilities: If (3)-(4) hold for all exchanges, then $X^* \in C$ is an optimal allocation $\iff$ the probabilities of execution shortfall are equal to specific thresholds:

$$
\mathbb{P}
\left(
M^* + \sum_{k=1}^{K} (\xi_k - Q_k)_+ - (\xi_k - Q_k - L^*_k)_+ < S
\right) = \frac{h + f + \lambda_o}{\lambda_u + \lambda_o}
$$

$$
\mathbb{P}
\left(
M^* + \sum_{k=1}^{K} (\xi_k - Q_k)_+ - (\xi_k - Q_k - L^*_k)_+ < S \bigg| \xi_j > Q_j + L^*_j
\right) = \frac{\lambda_o - (h + r_j)}{\lambda_u + \lambda_o}
$$

This gives another interpretation to parameters $\lambda_u, \lambda_o$: if trader specifies tolerance for execution risk in terms of maximal shortfall probability, then $\lambda_u, \lambda_o$ may be calibrated using the above.
Simple example

If \( \xi_1, \xi_2 \) are exponentially distributed with means \( \mu_1, \mu_2 \) respectively, then an optimal order allocation is given by:

\[
\begin{align*}
M^* &= Q_1 + Q_2 + S - z \\
L_1^* &= z - Q_1 + \mu_2 \log \left( \frac{\lambda_u + h + r_1}{\lambda_u + \lambda_o} \right) \\
L_2^* &= z - Q_2 + \mu_1 \log \left( \frac{\lambda_u + h + r_2}{\lambda_u + \lambda_o} \right),
\end{align*}
\]

where \( z \) is a solution of a transcendental equation. If \( \mu_1 = \mu_2 = \mu \), the equation:

\[
1 + \log \left( \frac{(\lambda_u + h + r_1)(\lambda_u + h + r_2)}{(\lambda_u + \lambda_o)^2} \right) + \frac{z}{\mu} = \frac{\lambda_u - (h + f)}{\lambda_u + \lambda_o} e^{\frac{z}{\mu}}
\]

Solution for \( \mu_1 \neq \mu_2 \) is given in our paper.
If $\xi_1, \xi_2$ are exponentially distributed with means $\mu_1, \mu_2$ respectively, then an optimal order allocation is given by:

\[
\begin{align*}
M^* &= Q_1 + Q_2 + S - z \\
L_1^* &= z - Q_1 + \mu_2 \log \left( \frac{\lambda_u + h + r_1}{\lambda_u + \lambda_o} \right) \\
L_2^* &= z - Q_2 + \mu_1 \log \left( \frac{\lambda_u + h + r_2}{\lambda_u + \lambda_o} \right),
\end{align*}
\]

(5)

where $z$ is a solution of a transcendental equation. If $\mu_1 = \mu_2 = \mu$, the equation:

\[
1 + \log \left( \frac{(\lambda_u + h + r_1)(\lambda_u + h + r_2)}{(\lambda_u + \lambda_o)^2} \right) + \frac{z}{\mu} = \frac{\lambda_u - (h + f)}{\lambda_u + \lambda_o} e^{\frac{z}{\mu}}
\]

Solution for $\mu_1 \neq \mu_2$ is given in our paper.

Some features of the single-exchange solution are present here, too:

- Pricing parameters affect the optimal allocation via a specific ratio.
- $L_{1,2}^*$ decrease with $Q_{1,2}$.
- $M^*$ increases with $S, Q_{1,2}$. 
Objective function is convex and has the form $V(X) = E[v(X, \xi)]$

To solve the problem in high dimensions we can use stochastic algorithms which approximate the gradient of $V$ along a random optimization path, e.g. Robbins–Monro, Nemirovskii et al. (2009)
Objective function is convex and has the form $V(X) = E[v(X, \xi)]$

To solve the problem in high dimensions we can use stochastic algorithms which approximate the gradient of $V$ along a random optimization path, e.g. Robbins–Monro, Nemirovskii et al. (2009)

Denote by $g(X, \xi) \triangleq \frac{\partial f(X, \xi)}{\partial X}$ a stochastic gradient. The algorithm is:
1: Set $X := X_0$, step size $\gamma$;
2: for $n = 1, \ldots, N$ do
3: Draw an independent random variable $\xi^n$ from distribution $F$
4: Set $X_n = X_{n-1} - \gamma g(X_{n-1}, \xi^n)$
5: end for
6: Set $\hat{X}^* := \frac{1}{N} \sum_{i=1}^{N} X_i$;

This does not require to know the form of the outflow distribution $F$ but only a way to sample from it!
The function $g(X_n, \xi)$ has a specific form in our problem:

$$g(X_n, \xi) = \begin{pmatrix}
  h + f - \lambda_u l_u(n) + \lambda_o l_o(n) \\
  - (h + r_1) l_1(n) - \lambda_u l_u(n) l_1(n) + \lambda_o l_o(n) l_1(n) \\
  \vdots \\
  - (h + r_K) l_K(n) - \lambda_u l_u(n) l_K(n) + \lambda_o l_o(n) l_K(n)
\end{pmatrix}$$

where $l_u(n) = \mathbb{1}_{\{A(X_n, \xi) < S\}}$, $l_o(n) = \mathbb{1}_{\{A(X_n, \xi) > S\}}$, $l_k(n) = \mathbb{1}_{\{\xi_k > Q_k + L_{k,n}\}}$

This iterative solution increases or decreases order sizes based on indicators $l_u(n)$, $l_o(n)$, $l_k(n)$ of order fills on previous iterations.

This approach does not rely on a parametric order flow model.

But it gives a specific procedure for re-sampling past order fills to obtain an estimate of an optimal order allocation.

In practice one can resample recent executions or recent bid queue outflow data, instead of using heuristics based on average fill statistics.
Figure: Convergence of numerical solution to an optimal point, in terms of

\[ W(X) = \frac{1}{L} \sum_{i=1}^{L} (C(X, \xi_i) + P(X, \xi_i)) \approx V(X) \] 

for different starting points.

Parameter values: \( S = 1000, Q = 2000, \xi \sim Pois(2200), h = 0.02, r = 0.002, f = 0.003, \lambda_o = 0.024, \lambda_u = 0.026. \)
Parameter values are same as on the previous figure, $Q_k = Q, r_k = r, k = 1, K, \xi_{n,k}$ are i.i.d. $X_M$ submits one market order, $X_L$ submits one limit order, $X_E$ submits all orders with sizes $S/(K + 1)$, $\hat{X}^*$ is an optimal solution.

<table>
<thead>
<tr>
<th>$K$</th>
<th>$(\hat{M}^<em>, \hat{L}_1^</em>, \hat{L}_2^<em>, \hat{L}_3^</em>, \hat{L}_4^*)/S$</th>
<th>$W(X_M)$</th>
<th>$W(X_L)$</th>
<th>$W(X_E)$</th>
<th>$W(\hat{X}^*)$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$S = 500$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>0.487 0.513</td>
<td>11.50</td>
<td>3.42</td>
<td>2.81</td>
<td>2.79</td>
</tr>
<tr>
<td>2</td>
<td>0.004 0.627 0.622</td>
<td>11.50</td>
<td>3.45</td>
<td>-2.84</td>
<td>-5.74</td>
</tr>
<tr>
<td>3</td>
<td>0.001 0.410 0.457 0.459</td>
<td>11.50</td>
<td>3.35</td>
<td>-5.25</td>
<td>-9.92</td>
</tr>
<tr>
<td>4</td>
<td>0.000 0.307 0.255 0.303 0.273</td>
<td>11.50</td>
<td>3.31</td>
<td>-6.45</td>
<td>-10.65</td>
</tr>
<tr>
<td></td>
<td>$S = 1000$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>0.736 0.264</td>
<td>23.00</td>
<td>16.34</td>
<td>14.84</td>
<td>14.22</td>
</tr>
<tr>
<td>2</td>
<td>0.512 0.332 0.350</td>
<td>23.00</td>
<td>16.48</td>
<td>-2.80</td>
<td>-2.58</td>
</tr>
<tr>
<td>3</td>
<td>0.288 0.356 0.358 0.341</td>
<td>23.00</td>
<td>16.49</td>
<td>-9.44</td>
<td>-11.34</td>
</tr>
<tr>
<td>4</td>
<td>0.063 0.368 0.343 0.350 0.347</td>
<td>23.00</td>
<td>16.43</td>
<td>-9.44</td>
<td>-11.34</td>
</tr>
<tr>
<td></td>
<td>$S = 5000$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>0.952 0.048</td>
<td>115.00</td>
<td>120.53</td>
<td>113.03</td>
<td>106.46</td>
</tr>
<tr>
<td>2</td>
<td>0.904 0.088 0.088</td>
<td>115.00</td>
<td>120.38</td>
<td>105.75</td>
<td>97.73</td>
</tr>
<tr>
<td>3</td>
<td>0.856 0.126 0.126 0.126</td>
<td>115.00</td>
<td>120.43</td>
<td>97.54</td>
<td>89.27</td>
</tr>
<tr>
<td>4</td>
<td>0.807 0.169 0.169 0.169</td>
<td>115.00</td>
<td>120.41</td>
<td>88.63</td>
<td>80.50</td>
</tr>
</tbody>
</table>

Table: Savings from optimal order routing.

Note that because order flows are i.i.d. it is optimal to oversize the total submitted quantity.
Figure: Sensitivity analysis for a numerical solution $\hat{X}^* = (M, L_1, L_2)$ with two exchanges and an optimal solution $(M^a, L^a)$ with the first exchange only.
Figure: Sensitivity analysis for a numerical solution $\hat{X}^* = (M, L_1, L_2)$ with two exchanges and an optimal solution $(M^a, L^a)$ with the first exchange only.
Figure: Sensitivity analysis for a numerical solution $\hat{X}^* = (M, L_1, L_2)$ with two exchanges and an optimal solution $(M^a, L^a)$ with the first exchange only.
Figure: Sensitivity analysis for a numerical solution $\hat{X}^* = (M, L_1, L_2)$ with two exchanges and an optimal solution $(M^a, L^a)$ with the first exchange only.
Practical example

How can we buy 10,000 shares of MSFT within one minute?

- Assume two exchanges are available - NASDAQ and BATS (Z).
- Average queue sizes are 12,392 and 8,179 shares respectively, average 1-minute volumes are 848 and 922 shares (in April 2012)
- Fees: 30 and 29 mills (1/10,000 of a dollar) per share
- Rebates: 20 and 25 mills per share, based on the lowest volume tier
- Half-spread: typically 50 mills
- Take 3 months of data (January - March 2012), separate observations into 3 equal-sized bins by initial queue size on NASDAQ and BATS, previous minute volume on NASDAQ and BATS (total $3^4 = 81$ bins) and apply the re-sampling algorithm.
Practical example

- Set $\lambda_u = \lambda_o = 0.01 = 2h$ and solve the problem separately for each bin using the re-sampling method.

- For illustration, solutions for “small” queues on NASDAQ and BATS, and “small” previous minute volume on NASDAQ:

<table>
<thead>
<tr>
<th>Previous BATS volume</th>
<th>$\hat{M}^*$</th>
<th>$\hat{L}_B^*$</th>
<th>$\hat{L}_N^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>low</td>
<td>9848</td>
<td>128</td>
<td>113</td>
</tr>
<tr>
<td>medium</td>
<td>9379</td>
<td>553</td>
<td>505</td>
</tr>
<tr>
<td>high</td>
<td>7458</td>
<td>2442</td>
<td>2301</td>
</tr>
</tbody>
</table>

- Then, use these solutions $\hat{X}^*$ for next month and compare with an equal split strategy $X_E = (3333, 3333, 3333)$:

<table>
<thead>
<tr>
<th>Average</th>
<th>$X_E$</th>
<th>$\hat{X}^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C(X, \xi)$, dollars</td>
<td>20</td>
<td>74</td>
</tr>
<tr>
<td>$P(X, \xi)$, dollars</td>
<td>58</td>
<td>4</td>
</tr>
<tr>
<td>$A(X, \xi)$, shares</td>
<td>4,179</td>
<td>9,625</td>
</tr>
</tbody>
</table>
Conclusion

- Unified treatment of the choice of order type (limit/market) and order routing decision across multiple trading venues as a convex optimization problem.
- Allows explicit choice of tradeoff between execution cost and execution risk.
- Case of a single exchange: reduces to an optimal choice of limit and market order sizes. ⇒ Explicit formulas for optimal limit and market order sizes.
- General case: efficient numerical solution.
- Simulation tests: using the proposed order placement policy substantially reduces transaction costs.
- Results consistent with intuitive insights.