Market Impact Paradoxes

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Abstract

The market impact (MI) of Volume Weighted Average Price (VWAP) orders is a slightly convex function of a trading rate, but most empirical estimates of transaction cost are concave functions. How is this possible? We suggest a model that fits all trading regimes and guarantees no-dynamic-arbitrage.
the number (all time/last year) of scholarly papers in SSRN with the phrase

- ‘market impact’ - 291(58)
- ‘price impact’ - 578(68)
- ‘transaction cost’ - 1047(101)

in title, abstract, or keywords
Understanding market impact is important for several reasons.

- **One motivation is practical:** To know whether a trade will be profitable it is essential to be able to estimate transaction costs, and in order to optimize a trading strategy to minimize such costs, it is necessary to understand the functional form of market impact.

- **Another motivation is ecological:** Impact exerts selection pressure against a fund becoming too large, and therefore is potentially important in determining the size distribution of funds.

- **Finally, an important motivation is theoretical:** Market impact reflects the shape of excess demand, the understanding of which has been a central problem in economics since the time of Alfred Marshall.

Farmer et al. (2013) [10]
In financial markets, market impact is the effect that a market participant has when it buys or sells an asset. It is the extent to which the buying or selling moves the price against the buyer or seller, i.e. upward when buying and downward when selling. It is closely related to market liquidity; in many cases ‘liquidity’ and ‘market impact’ are synonymous.

From Wikipedia, the free encyclopedia
Impact can be

- temporary
- permanent
- full
- immediate
- instantaneous
- impulse
- total

Related terms are implementation shortfall and slippage.
Grinold and Kahn (1999) and Almgren and Chriss (1999) (GKAC) independently pioneered application of calculus of variation to the problem of portfolio liquidation. Today most modern trading engines use different modifications of their method. mean-variance utility

\[
\Phi = \int_0^t (E(R) - \tilde{\lambda} Var(R))dt
\]

\[
\Phi[x] = \int_0^T (h[x]\dot{x} - \tilde{\lambda}(Sx\sigma)^2)dt
\]  \hspace{1cm} (1)

\[
h[x] = \int_0^t f(q(\tau))K(t - \tau)dt
\]  \hspace{1cm} (2)
In linear case Euler equation

\[ 2k^2x = \frac{d}{dt} \int_0^T \dot{x}(\tau)K(|t - \tau|)d\tau \]  

(3)

with a kernel \( K(t) = \delta(t) \) has an analytical solution

\[ x(t) = X_0 \frac{\sinh (k(T - t))}{\sinh (kT)} + X_T \frac{\sinh (kt)}{\sinh (kT)} \]  

(4)
$R$ is an absolute asset price return in $\$$

$Var$ is a variance

$T$ is a time horizon.

$X_0$ and $X_T$ are initial and terminal position in a stock

$x$ is the current position

$\dot{x}$ is its time derivative

trading rate $q = -\dot{x}$ is positive when cash flow goes in

$\eta$ is a coefficient of temporary market impact

$\tilde{\eta} = \eta \frac{\sigma}{ADV_{S_0}}$

$ADV$ is an Average Daily Volume

$\tilde{\lambda}$ is a risk-averse parameter.

$\lambda = k^2 = \tilde{\lambda}/\tilde{\eta}$

$f(q)$ is reduced to a linear function of trading rate, $f = -\tilde{\eta}\dot{x}$
Exponential Kernel

Problems with $\delta$ - function kernel

- it cannot describe the market impact of a single discrete trade
- instantaneous recovery assumption is unrealistic

The next step after Dirac’s delta function is the exponential kernel

$$K(t) \sim \exp(-\beta t)$$

The general solution for the optimal trajectories would be given by

$$x = C_1 e^{kt} + C_2 e^{-kt} + D_1 \mathcal{H}(t) + D_2 \mathcal{H}(t - T) \quad (5)$$

$\mathcal{H}(t)$ is a Heavyside’s function.
We need four equations to find four arbitrary constants. Two equations represent initial and terminal conditions and two additional equations follow from the requirement that $x(t)$ doesn’t have $\exp(\pm \beta t)$ terms. After some simple but tedious algebra we get a solution in a familiar form:
Pret a manger solution for optimal trajectory

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\[
 x(t) = X_0 B \frac{\sinh \left( (k(T - t) + A) \right)}{\sinh \left( kT + 2A \right)} + X_T B \frac{\sinh \left( kT \right)}{\sinh \left( kT + 2A \right)}
\]

where

\[
 A = \ln \sqrt{\frac{\beta + k}{\beta - k}}, \quad B = \frac{k}{\sqrt{\lambda}}, \quad k^2 = \frac{\lambda \beta^2}{\lambda + \beta^2}
\]

(6)

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More detailed derivation and more general solution can be found in Skachkov [21]. For risk-neutral traders ($\lambda \to 0$) the optimal schedule under exponential impact relaxation is a combination of two jumps and straight line between them.
Introduction
Analytical Models
Diffusion Model
Paradoxes
Results and discussion

More detailed derivation and more general solution can be found in Skachkov [21]. For risk-neutral traders ($\lambda \to 0$) the optimal schedule under exponential impact relaxation is a combination of two jumps and straight line between them.

$$\lim_{\lambda \to 0} x = (X_0 - \Delta X_0) \frac{T - t}{T} + (X_T + \Delta X_T) \frac{t}{T}$$

$$\Delta X_0 = \Delta X_T = \frac{X_0 - X_T}{\beta T + 2}$$

Optimal trading strategy with the exponential kernel was the subject of Obizhaeva and Wang (2005) study [20]. They were the first to point out that discontinuity of optimal paths at the ends of a time interval and to derive optimal risk-neutral trajectories (7). With $\beta \to \infty$ our result (6) goes to the classic solution (4).
Diffusion Model

This is a scheme of a drill stem test.

Drawdown and buildup tests
\[ f_t - \kappa f_{xx} = 0, \quad 0 < x < x_2 \]
\[ f(x, t) = h(t), \quad -x_1 < x < 0 \]

where \( \kappa \) is a diffusion coefficient.

\[ f(0, x) = 0 \]
\[ h(t) = f(t, 0) \]
\[ c \cdot h_t = q(t) + \kappa f_x, \quad x = 0 \]
\[ f_x = 0 \quad (\text{impenetrable wall at } x = x_2) \]
The solution of the system (9) in Laplace domain

\[ \bar{h} = \frac{\bar{q}}{cs + \kappa \sqrt{\frac{s}{\kappa}} \tanh(\sqrt{\frac{s}{\kappa}} \cdot x_2)} = \bar{q} \bar{K} \]  

(10)

the solution (10) in unbounded area can be analytically inverted back to time domain

\[ \bar{h} = \frac{\bar{q}}{cs + \sqrt{\kappa s}} = \bar{q} \bar{K} \]  

(11)

\[ h_\delta(t) = K(t) = c^{-1} \cdot \exp(\tilde{t}) \cdot \text{erfc}(\sqrt{\tilde{t}}), \quad \tilde{t} = \frac{\kappa}{c^2} t \]  

(12)
No-dynamic-arbitrage

About trading triggered price manipulations see Huberman and Stanzl, (2004), [17], Alfonsi, Schied, Slynko (2012) [1], Gatheral and Schied (2013), [12]

\[
\Delta W = \int_{V_0}^{V(T)} P \, dV(t)
\]  \hspace{1cm} (13)

- $P$ is a pressure (expected stock price)
- $V(t)$ is a volume (current volume traded).

According to the first law of thermodynamics

\[
\Delta W = \oint P \, dV(t) \geq 0
\]  \hspace{1cm} (14)
The first law of Mercatodynamics

For a thermodynamic cycle of a closed system, which returns to its original state, the heat $Q_{in}$ supplied to a closed system in one stage of the cycle, minus that $Q_{out}$ removed from it in another stage of the cycle, equals the net work done by the system...

For a mercatodynamic cycle, the wealth supplied to a closed system, minus that removed from it, equals the net payment made by the system. It is not possible to construct a perpetuum mobile machine which will continuously trade without consuming wealth.
As a free gift from the integral transform approach we can get the required rate with the given price.

For example, if we want to keep the constant price difference with arrival price $\Delta S$, we need after initial shot ($q$ is Dirac delta function at the initial time) trade with the decreasing as square root of time rate.

\[
q = c\delta(t) + \sqrt{\frac{\kappa}{\pi t}}
\]  
(15)

Correspondingly, to sustain $S \propto t^\alpha$ price growth, trading rate

\[
q = c \cdot \alpha t^{\alpha-1} + \sqrt{\kappa} t^{\alpha-\frac{1}{2}} \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha + \frac{1}{2})}
\]  
(16)

is required.
Diffusion model is rich and elaborated, has a huge library of the problems in physics and engineering, that were solved and analyzed in details: namely in classic theory of conduction of heat in solids and in porous media hydrodynamics and modern oil and gas well test engineering.

More complex models:

- ‘Skin effect’
- Dual porosity (fractures - matrix) formation
- Nonlinear problems of real gas pseudo-pressure dynamics
- Arbitrary dimension diffusion
Bad news and a good news

We derived a comparatively simple linear model that is dynamic arbitrage proof and correctly describes decay of market perturbations.

**Bad news:**

- Numerous empirical evidences that market impact is a concave function of a trading rate
- \( VWAP \) algorithm performance is flat if market participation is small \((< 1\%)\) and then slightly convex \([8]\) up to 50\% \(ADV\).
Bad news and a good news

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**Good news:**

- Those evidences contradict each other.
Limit Order Book (LOB) and Instantaneous MI

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Instantaneous impacts. Convex or Concave?

- ‘Unbiased‘ instantaneous price impact should have a complex shape: first concave, then convex.
- Convexity $\leq$ human error/SW bug or market manipulation, e.g Stop Loss Hunter and Whack-A-Mole strategies.
- Instantaneous price impact is equal to zero, otherwise it is equal to 1 tick for liquid stocks [13].
- Instantaneous market impact is a linear function of order flow imbalance, Cont, Kukanov and Stoikov (2012) [9].
- Orders do not impact prices. It is more accurate to say that orders forecast prices. Hasbrouck (2007)
- Implementation shortfall is not equal to zero.
MI parameters calibration

- Extracting MI parameters from market data is a job that never guarantees conclusive results: the samples cannot be representative without access to proprietary information, data is not clean enough and some filtration with additional assumptions is needed, e.g. for relaxation time.

- Data preprocessing is never neutral: it filters out ‘biased’ (actually conditioned to intelligence of the engine and a trader who monitors it) and favors mechanical style execution.
Efficient Trading Hypothesis (ETH)

Algorithmic trading is efficient

Corollary

1. Parent orders are optimally split into child orders.

Corollary

2. The implementation shortfall has the minimum: it is the best transaction cost per share that can be achieved in one sided trading.
In empirical studies it would be easy to take linear function for slowly growing concave functions, e.g., power or even logarithmic, especially when additional point $(0, 0)$ is tacitly assumed. Note, that $R^2$ for model parameters calibrations in this case is less than 1%.
Evidence of market impact concavity

trading rule of thumb (from Grinold and Kahn (1999)) and empirical results by Almgren et al. (2005)

- it costs roughly one day volatility to trade one day’s volume.

$$\Delta W = C_1 + C_2 \cdot \sigma \cdot \sqrt{\frac{Q}{ADV}}$$ (17)

- Almgren et al. (2005) [4] analyzed almost 700,000 (29,509) US stock trade orders. Their empirical studies result in the similar power law.

$$h = \eta \cdot \sigma \cdot \text{sgn}(q) \cdot \left| \frac{Q}{ADV \cdot T} \right|^\beta = \eta \cdot \sigma \cdot \text{sgn}(q) \cdot \left| \frac{q}{ADV} \right|^\beta, \beta \approx 0.6$$ (18)
Conventional wisdom and rigorous data mining both suggest a similar concave dependence of market impact on trading volume.

With a day’s volume we are in a continuous trading regime.

It seems that our attempt to reconcile linear instantaneous impact and concave impact of continuous trading failed.

To understand what happened, let us look at the more general equations.
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To understand what happened, let us look at the more general equations.

\[
\Delta W = \eta \frac{f(q)q}{Q} \int_0^T dt \int_0^t K(\tau)d\tau = \eta \frac{f(q)}{T} K_{-2}(T) \quad (19)
\]

In GKAC model \( K(t) = \delta(t), \ K_{-1}(t) = 1, \ K_{-2}(t) = t \) and

\[
\Delta W = \eta \cdot f(q)
\]
The *GKAC* model temporary impact depends only on trading rate. The specific numerical examples of permanent and temporary impact costs for two large-cap stocks were shown in (Table 3) of [4]. The execution of 10% of *ADV* shares was completed in 0.5, 0.2 and 0.1 days. Temporary impact of both stocks purchases follows the law (18). Those examples directly state that

\[ \Delta W \propto \sigma \cdot T^{-\beta} \mid Q=const \propto \sigma \cdot q^\beta \mid Q=const \]  

(20)
Grinold & Kahn [14] give an elegant heuristic derivation of this equation (17), (chapter 16, Equation 16.4). They explain that liquidation time is proportional to the size of stock inventory (chapter 16, Equation 16.1). In our notation

$$T \propto \frac{Q}{ADV} \quad (21)$$

Substituting into (17), we get

$$\Delta W - C_1 \propto \sigma \cdot \sqrt{T} \quad (22)$$
Plugging an asymptotic form of a diffusion kernel (12) into a general equation for the implementation shortfall (19)

\[ \Delta W = \eta \cdot \frac{q}{T} K_{-2}(T) = \tilde{\eta} \cdot \frac{1}{\sqrt{\kappa}} \cdot q \cdot \sqrt{T}, \quad T \gg 1 \quad (23) \]

Comparing equations (23) and (17), we found the meaning of diffusion coefficient \( \kappa \) in our ‘information space’. This parameter controls the speed of market response and, therefore, controls the volatility of a stock.

\[ \sigma \sim \frac{1}{\sqrt{\kappa}} \quad (24) \]
Finally we get a law for all three regimes: if parent order is big enough for continuous rate approximation and doesn’t exceed critical value that can crash the market

\[ \sigma \cdot \sqrt{T} \cdot q = \sigma \cdot \frac{Q}{\sqrt{T}} = \sigma \cdot \sqrt{Q} \cdot \sqrt{q} = C \cdot \Delta W \quad (25) \]

For continuous and elastic trading: isochronic (trading time \( T = \text{const} \)) market impact is linear on trading rate, isochoric (trading volume \( Q = \text{const} \)) market impact is proportional to the square root of trading rate \( q \), and isotachic (trading rate \( q = \text{const} \)) market impact is proportional to the square root of trading volume \( Q \).
Noticing that $\sigma \cdot \sqrt{T}$ is a volatility of stock $\sigma_T$ evaluated for period of time $0 < t < T$, we can rewrite (25) in a more balanced and concise form

$$\frac{\Delta W}{\sigma_T \cdot q} = C$$  \hspace{1cm} (26)
Discrete and Continuous Trading. Isochronic regime.

Isochronic regime, i.e. trading time $T$ is constant $T = 0.5$

Loglog plot for trading cost versus trading rate

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Discrete and Continuous Trading II. Isochoric regime.

trading volume
\[ Q = 1.2 \cdot e^{-3} ADV \]
is constant.

Loglog plot for trading cost versus trading rate
The diffusion kernel presented in this paper explains many empirical market impact estimations and allows square root \textit{metaorder market impact} and linear \textit{instantaneous market impact} to coexist.

We consider rehabilitation of a linear impact model as one of the main results of this paper.

\textbf{marginally impact for strictly sublinear power functions is singular at } q = 0.

Market impact and all market movements reflect information flow. We assume that dissemination of information is a diffusion rather than instantaneous process.
The diffusion model satisfies the no-dynamic-arbitrage principle and can explain empirical results for the all regimes of trading:

- **Isochronic** (constant time - various volume and rate) market impact cost is a linear function of trading rate.
- **Isochoric** (constant volume - various time and rate) market impact cost is a square root function of trading rate.
- **Isotachic or isokinetic** (constant rate - various time and volume) market impact cost is a square root function of trading volume.
We didn’t touch many important aspects of market impact theory in this paper, e.g., relation between temporary and permanent impact, Efficient market Hypothesis, fair pricing condition\(^1\) [10] and supply-demand balance in general.

After the work by Kyle \([19]\) (1985) it is common to describe market dynamics as a contest of three parties: a single insider who has unique knowledge of ‘fair’ price, noise traders who trade randomly; and market makers who set the prices conditional on trading flow\(^2\).

\(^{1}\text{Under fair pricing the average execution price is equal to the final price.}\)

\(^{2}\text{(FGLW) [10] modified the first agent assuming large number of informed traders with the same long term return prediction.}\)
The typical signal of informed long term investors is not interesting for intraday traders because its daily information ratio $\ll 1$.

Market makers try to close all their positions by the end of the day and the retail noisy traders tend to trade following the market.

This is another paradox: it is not clear, who is going to take open positions overnight and why.

One can be only sure of “the subtle nature of ‘random’ price changes” [6] and the subtle nature of the other market laws and hypotheses.
I would like to thank all people I ever discussed the exiting problems of market microstructure with, especially my former coworkers Mark Bamber, Boris Drovetsky, and Denis Ignatovich.


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